

Odderon with a running coupling constant

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Abstract. The running coupling is introduced into the equation for the odderon via the bootstrap relation. It is shown that the previously found odderon state with a maximal intercept, which is constructed from an antisymmetric pomeron wave function, continues to exist in the running coupling case. Its intercept is found to remain equal to unity independent of the behaviour assumed for the running coupling at low momenta.

1 Introduction

Lately renewed interest has been shown in the introduction of a running coupling into the BFKL dynamics. The summation of contributions from quark–antiquark loops to the evolution of the gluon density has been used to restore the full dependence of the coupling on the running scale in the color dipole approach [1–3]. It turned out that the obtained kernel for the linear BFKL equation essentially coincides with the one which we found many years ago by imposing the bootstrap relation necessary for the fulfillment of unitarity [4, 5]. In this paper we draw attention to the fact that the bootstrap relation in fact allows one to derive also the structure of the odderon in the perturbative QCD approach. It is of particular interest to know what happens to the odderon state with a maximal intercept unity found in [6] and expressed via an antisymmetric pomeron state when a running coupling is introduced. We demonstrate that since the bootstrap relation plays a decisive role in construction of this solution to the odderon equation, the introduction of the running coupling via the bootstrap allows one to preserve this particular solution.

As in our earlier papers, we have to stress from the start that the introduction of the running coupling into the BFKL formalism cannot be made rigorously and uniquely. The formalism admits transverse momenta of any magnitude, including very small ones, for which the concept of the gluon and its coupling loses any sense. The introduced running coupling has to be artificially continued to small momentum values, where it is completely undetermined. As a result the actual values for the pomeron and hence odderon intercepts generally depend on a chosen behaviour of the running coupling in the region of very low momenta. However the particular odderon states constructed from antisymmetric pomeron states continue to have their maximal intercept exactly equal to unity, as in the fixed coupling case studied in [6].

Note that we use the bootstrap equation only in the lowest order of the coupling. So one can hope for only establishing the leading order behaviour in the running coupling. For the pomeron, in the next-to-leading order, it has been explicitly found that the bootstrap method correctly reproduces the part of the kernel responsible for the running of the coupling but not the rest piece [8–10]. One expects the same to be true also for the odderon.

We limit ourselves to the case of the odderon at rest (with zero total momentum). Generalization to non-zero total momenta is straightforward.

2 The running coupling from the bootstrap

As mentioned in the introduction, our idea to introduce a running coupling via the bootstrap was presented in [4, 5]. It consists in expressing both the gluon trajectory and intergluon interaction in terms of a single function $\eta(q)$ of the gluon momentum in the way that preserves the bootstrap relation. The function $\eta(q)$ then can be chosen to conform to the high-momentum behaviour of the gluon density with a running coupling. Let the two-gluon equation be written as an inhomogeneous Schrodinger equation,

$$(H - E)\psi = F, \quad (1)$$

with the Hamiltonian for the colour group $SU(N_c)$

$$H = \frac{1}{2}N_c(t(q_1) + t(q_2)) + (T_1 T_2)V. \quad (2)$$

Here $(N_c/2)t(q) \equiv -\omega(q)$ is the kinetic energy of the gluon given by its Regge trajectory with the minus sign. We write $t(q)$ in the form

$$t(q) = \int \frac{d^2q_1}{(2\pi)^2} \frac{\eta(q)}{\eta(q_1)\eta(q - q_1)}, \quad (3)$$

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where $\eta(q)$ is an arbitrary function satisfying the condition

$$\eta(0) = 0, \quad (4)$$

which guarantees that the gluon trajectory passes through zero at $q = 0$ in accordance with the properties of the gluon. The interaction term, apart from the product of the gluon colour vectors T_i^a , $i = 1, 2$, $a = 1, \dots, N_c$, involves the interaction kernel V that we also express via the same function $\eta(q)$:

$$V(q_1, q_2 | q'_1, q'_2) = \frac{\eta(q_1 + q_2)}{\eta(q'_1)\eta(q'_2)} - \left(\frac{\eta(q_1)}{\eta(q'_1)} + \frac{\eta(q_2)}{\eta(q'_2)} \right) \frac{1}{\eta(q_1 - q'_1)}. \quad (5)$$

It conserves momentum: $q_1 + q_2 = q'_1 + q'_2$. Comparing (3) with (5) one finds the bootstrap relation, satisfied for arbitrary $\eta(q)$:

$$\int \frac{d^2 q_1}{(2\pi)^2} V(q_1, q_2 | q'_1, q'_2) = t(q_1) + t(q_2) - t(q_1 + q_2). \quad (6)$$

From the high-momentum behaviour of the gluon distribution with a running coupling one finds

$$\eta(q) = \frac{1}{2\pi} b q^2 \ln \frac{q^2}{\Lambda^2}, \quad q^2 \gg \Lambda^2, \quad (7)$$

where Λ is the standard QCD parameter and

$$b = \frac{1}{12} (11N_c - 2N_f). \quad (8)$$

The asymptotic (7) and condition (4) are the only properties of $\eta(q)$ that follow from the theoretical reasoning. A concrete form of $\eta(q)$ interpolating between (4) and (7) may be chosen differently. One hopes that physical results will not too strongly depend on the choice.

The fixed coupling case corresponds to the choice

$$\eta(q) = \frac{2\pi}{g^2} q^2. \quad (9)$$

Then one finds the standard BFKL expressions for the trajectory $t(q)$ and the interaction $V(q_1, q_2 | q'_1, q'_2)$.

It is customary to discuss a homogeneous equation (1) (with $F = 0$) to seek the ground state energy, which determines the rightmost singularity in the complex angular momentum j related to the energy by

$$j = 1 - E. \quad (10)$$

For our purpose it is, however, essential to conserve the inhomogeneous term $F(q_1, q_2)$, which represents the two-gluon-particle vertex.

Consider the vector colour channel with $T_1 T_2 = -N_c/2$ and, most important, assume that the inhomogeneous term depends only on the total momentum of the two gluons: $F = F(q_1 + q_2)$. Then using the bootstrap relation (6) one easily finds the solution to (1):

$$\psi(q_1, q_2) = \psi(q_1 + q_2) = \frac{F(q_1 + q_2)}{\omega(q_1 + q_2) - E}. \quad (11)$$

It means that the two gluons 1 and 2 have fused into a single one with the momentum $q_1 + q_2$. This phenomenon is true for arbitrary $\eta(q)$.

3 The odderon

3.1 The bootstrap for three gluons

Now we generalize the bootstrap for two gluons to the case of three gluons, relevant for the odderon. For three gluons (1) holds with the Hamiltonian that is a sum of kinetic energies and pair interactions:

$$H = \frac{1}{2} N_c \sum_{i=1}^3 t(q_i) + \sum_{i < k}^3 (T_i T_k) V_{ik}. \quad (12)$$

Here V_{ik} is the interaction of gluons i and k with the kernel $V(q_i, q_k | q'_i, q'_k)$. We do not impose any restrictions on the total colour $T = \sum_{i=1}^3 T_i$. (However only for $T = 0$ the Hamiltonian is infrared stable.)

Assume now that $T_1 T_2 = -N_c/2$, i.e. the gluons 1 and 2 form a colour vector. Then we demonstrate that for a certain specific choice of the inhomogeneous term F , (1) for three gluons reduces to that for two gluons, the gluons 1 and 2 fused into a single gluon, which carries their total momentum and colour. In other words, as in the two-gluon case, a pair of gluons in the adjoint representation is equivalent to a single gluon.

Of course, the specific form of the inhomogeneous term is a decisive instrument for bootstrapping the two gluons 1 and 2. For three gluons we choose

$$F_3(q_1, q_2, q_3) = \int \frac{d^2 q'_3}{(2\pi)^2} \widehat{W}(q_1, q_2, q_3 | q'_1, q'_3) \psi_2(q'_1, q'_3) + F_2(q_1 + q_2, q_3), \quad (13)$$

with $q'_1 + q'_3 = q_1 + q_2 + q_3$. In (13) ψ_2 is a solution of the Schroedinger equation (1) with the inhomogeneous term F_2 . Gluon 1 in it substitutes the fused initial gluons 1 and 2. The kernel \widehat{W} is an operator acting also on colour indices of ψ_2 . It is convenient to retain a pair of colour indices for gluon 1 in ψ_2 inherited from the initial gluons 1 and 2 by means of a projector onto the adjoint representation in the colour $T_1 + T_2$. This allows one to consider \widehat{W} as an operator acting in the colour space of three gluons (not two). Then it has the form

$$\widehat{W}(q_1, q_2, q_3 | q'_1, q'_i) = (T_1 T_3) W(q_2, q_1, q_3 | q'_1, q'_3) + (T_2 T_3) W(q_1, q_2, q_3 | q'_1, q'_3), \quad (14)$$

where the momentum space kernel W is a difference between two kernels (5), expressed via function $\eta(q)$:

$$W(q_2, q_1, q_3 | q'_1, q'_3) = V(q_1, q_3 | q'_1 - q_2, q'_3) - V(q_1 + q_2, q_3 | q'_1, q'_3). \quad (15)$$

The explicit form of W is irrelevant for our purpose. In terms of function $\eta(q)$ it can be found in [11]. It possesses a symmetry property, which is important for us:

$$W(q_1, q_2, q_3|q'_1, q'_3) = W(q_3, q_2, q_1|q'_3, q'_1). \quad (16)$$

Note that in the fixed coupling case this kernel reduces to the Bartels kernel $K_{2 \rightarrow 3}$, which describes the transition from two BFKL pomerons to three [12].

We are going to show that, with the inhomogeneous term F_3 given by (13) and $T_1 T_2 = -N_c/2$, the Schroedinger equation (1) for three gluons is solved by

$$\psi_3(q_1, q_2, q_3) = \psi_2(q_1 + q_2, q_3). \quad (17)$$

Indeed putting (17) into the equation we find that the interaction term $(T_1 T_2) V_{12}$, according to (6), substitutes the sum of kinetic terms $t(q_1) + t(q_2)$ for the gluons 1 and 2 by $t(q_1 + q_2)$, which is precisely the kinetic term for the function $\psi_2(q_1 + q_2)$. The interaction of gluon 1 with the third one takes the form

$$\begin{aligned} (T_1 T_3) \int \frac{d^2 q'_3}{(2\pi)^2} V(q_1, q_3|q'_1, q'_3) \psi_2(q'_1 + q_2, q'_3) \\ = (T_1 T_3) \int \frac{d^2 q'_3}{(2\pi)^2} V(q_1, q_3|q'_1 - q_2, q'_3) \psi_2(q'_1, q'_3). \end{aligned} \quad (18)$$

The momentum is conserved during the interaction, so that on the left-hand side $q_1 + q_3 = q'_1 + q'_3$ and on the right-hand side $q_1 + q_2 + q_3 = q'_1 + q'_3$. The term (18) is cancelled by the identical contribution with the opposite sign coming from the inhomogeneous term F_3 (the first term in (15) in the part of \widehat{W} proportional to $T_1 T_3$). Instead of it from the inhomogeneous term comes the contribution (the second term in (15))

$$(T_1 T_3) \int \frac{d^2 q'_3}{(2\pi)^2} V(q_1 + q_2, q_3|q'_1, q'_3) \psi_2(q'_1, q'_3). \quad (19)$$

In the same manner the interaction of gluon 2 with the third one,

$$\begin{aligned} (T_2 T_3) \int \frac{d^2 q'_3}{(2\pi)^2} V(q_2, q_3|q'_2, q'_3) \psi_2(q_1 + q'_2, q'_3) \\ = (T_2 T_i) \int \frac{d^2 q'_3}{(2\pi)^2} V(q_2, q_3, q'_2 - q_1, q'_3) \psi_2(q'_2, q'_3), \end{aligned} \quad (20)$$

is transformed by the term in F_3 proportional to $T_2 T_3$ into

$$(T_2 T_3) \int \frac{d^2 q'_3}{(2\pi)^2} V(q_1 + q_2, q_3|q'_1, q'_3) \psi_2(q'_1, q'_3). \quad (21)$$

The two terms (18) and (21) sum into

$$(T_1 + T_2, T_3) \int \frac{d^2 q'_3}{(2\pi)^2} V(q_1 + q_2, q_3|q'_1, q'_3) \psi_2(q'_1, q'_3), \quad (22)$$

which is precisely the correct form for the interaction of the gluon with the total colour $T_1 + T_2$ and momentum $q_1 + q_2$ in which the initial gluons 1 and 2 have fused. As a result we obtain the Schroedinger equation (1) for the function ψ_2 describing two gluons with the inhomogeneous term F_2 .

The inhomogeneous term of the form (13) actually appears in the theory of three reggeized gluons, provided one takes into account the possibility for transitions from 2 to 3 gluons. In the case of the fixed coupling constant this fact was discovered in the course of the perturbative analysis of the system of four reggeized gluons in the triple Regge kinematical region [13, 14]. Introducing a running coupling via the bootstrap allows one to preserve this result for a general choice of the function $\eta(q)$.

We stress that this result is valid for any value of the total colour of the three-gluon system and any choice of the symmetry of the spatial wave function. In the particular case of the colorless three-gluon system with a d -structure of the color wave function (the odderon state) the bootstrap allows us to find a specific solution to the odderon equation, expressed via an antisymmetric pomeron state.

3.2 A solution of the odderon equation

In the case of the odderon the total colour is zero, so that any pair of gluons is in the colour vector state and $T_i T_k = -N_c/2$ for $i < k = 1, 2, 3$. The total color wave function is $d_{a_1 a_2 a_3}$ where a_i is the color of the i th reggeized gluon. Separating this color factor one finds the equation for the odderon momentum wave function:

$$(H_{\text{od}} - E) \psi_{\text{od}} = 0, \quad (23)$$

with a Hamiltonian

$$H_{\text{od}} = \frac{1}{2} N_c \left(\sum_{i=1}^3 t(q_i) - \sum_{i < k}^3 V_{ik} \right). \quad (24)$$

Now we use the bootstrap results obtained above. Consider the odderon equation with the inhomogeneous term (13) in which $T_1 T_3 = T_1 T_2 = -N_c/2$, the colour wave function $d_{a_1 a_2 a_3}$ and $F_2 = 0$. Our result tells us that this equation is solved by the pomeron momentum wave function

$$\psi^{(12)} = \psi_2(q_1 + q_2, q_3), \quad (25)$$

which satisfies the homogeneous Schroedinger equation (1) with Hamiltonian (2) in which $T_1 T_2 = -N_c$ (the BFKL equation in the case of a fixed coupling). Explicitly we find that in momentum space

$$(H_{\text{od}} - E) \psi^{(12)} = F_3^{(12)}, \quad (26)$$

where

$$\begin{aligned} F_3^{(12)}(1, 2, 3) = -\frac{1}{2} N_c \\ \times \int \frac{d^2 q'_3}{(2\pi)^2} (W(2, 1, 3|1', 3') + W(1, 2, 3|1', 3')) \psi_2(1', 3'). \end{aligned} \quad (27)$$

Here and in the following for brevity we denote gluon momenta just by numbers: $1 \equiv q_1$, $1' \equiv q_1'$ etc.

Next we cyclically permute the gluons 1–3 to obtain two more relations:

$$(H_{\text{od}} - E)\psi^{(23)} = F_3^{(23)} \quad (28)$$

and

$$(H_{\text{od}} - E)\psi^{(31)} = F_3^{(31)}, \quad (29)$$

where, for instance,

$$\psi^{(23)} = \psi_2(q_2 + q_3, q_1) \quad (30)$$

and

$$F_3^{(23)}(1, 2, 3) = -\frac{1}{2}N_c \times \int \frac{d^2 q_3'}{(2\pi)^2} (W(3, 2, 1|1', 3') + W(2, 3, 1|1', 3')) \psi_2(1', 3') \quad (31)$$

(the integration momenta q_3' and $q_1' = q_1 + q_2 + q_3 - q_3'$ do not change under permutations of the external momenta). We add all the three Schroedinger equations obtained in this manner together to obtain

$$(H_{\text{od}} - E)(\psi^{(12)} + \psi^{(23)} + \psi^{(31)}) = F_3^{(12)} + F_3^{(23)} + F_3^{(31)} \equiv F_3^{\text{tot}}. \quad (32)$$

The total inhomogeneous term has the structure

$$F_3^{\text{tot}}(1, 2, 3) = -\frac{N_c}{2} \int \frac{d^2 q_3'}{(2\pi)^2} U(1, 2, 3|1', 3') \psi_2(1', 3'), \quad (33)$$

where

$$U(1, 2, 3|1', 3') = W(2, 1, 3|1', 3') + W(1, 2, 3|1', 3') + W(3, 2, 1|1', 3') + W(2, 3, 1|1', 3') + W(1, 3, 2|1', 3') + W(3, 1, 2|1', 3'). \quad (34)$$

Using property (16) one easily finds that U is symmetric in the last pair of arguments:

$$U(1, 2, 3|1', 3') = U(1, 2, 3|3', 1'). \quad (35)$$

It follows that if function $\psi_2(q_1, q_2)$ is antisymmetric in its variables, the total inhomogeneous term (33) in (32) vanishes and the sum

$$\psi_{\text{od}} = \psi^{(12)} + \psi^{(23)} + \psi^{(31)} \quad (36)$$

is a solution of the odderon equation

$$(H_{\text{od}} - E)\psi_{\text{od}} = 0. \quad (37)$$

Thus from any antisymmetric solution of the pomeron equation one gets a corresponding solution of the odderon equation.

For a fixed coupling this result was found in [6]. Our derivation generalizes it to the case of a running coupling, provided it is introduced in the manner which preserves the bootstrap relations.

4 The pomeron and odderon intercepts

With the running coupling constant the intercept of the pomeron states generally begins to depend on the way in which the constant is chosen to behave at small values of momenta (where in fact it is not defined physically). We recall that our function $\eta(q)$ is fixed only by its asymptotic behaviour at large values of q (7) and its value at $q = 0$ (4). A possible simple form of $\eta(q)$ satisfying these conditions and no more singular at $q = 0$ than in the fixed coupling case is

$$\eta(q) = \frac{1}{2\pi} b q^2 \ln \left(\frac{q^2}{\Lambda^2} + \xi^2 \right), \quad (38)$$

with two parameters, the standard QCD parameter Λ and $\xi > 1$, constrained by the experimental value of the coupling constant $\alpha_s(q^2)$ at some chosen value of q^2 . This form implies that the running coupling $\alpha_s(q)$ steadily grows towards small values of q and freezes at $q \sim q_0 \sim \xi \Lambda$ having the value

$$\alpha_s(0) = \frac{\pi}{2b \ln \xi} \quad (39)$$

at $q = 0$. Choosing m_w as the experimental scale and taking $\alpha_s(m_w^2) = 0.126$ (to have $\Lambda \simeq 0.2$ GeV/ c at small values of ξ , in the one-loop approximation) we find that the interval of possible values of ξ is $1 \leq \xi \leq \xi_{\text{max}} = 1.57 \times 10^5$. As ξ rises, so do Λ and q_0 , whereas $\alpha_s(0)$ and the intercept Δ steadily diminish towards their limiting values at ξ_{max} , 0.126 and 0.330 respectively. Values of all these quantities at different ξ are presented in Table 1.

In this respect it is worth noting that the introduction of the running coupling does not eliminate the arbitrariness in the value chosen for the coupling in the fixed coupling case, just trading this arbitrariness for the one in the choice of, say, $\alpha_s(0)$.

However, it is remarkable that the introduction of the running coupling does not change the maximal intercept of the odderon, which, with unity subtracted, remains equal to zero. In the fixed coupling case the maximal intercept

Table 1. Pomeron intercepts

ξ	Λ (GeV/ c)	$\alpha_s(0)$	q_0 (GeV/ c)	Δ
2.0	0.197	2.18	0.278	5.69
2.0×10^2	0.197	0.284	2.78	0.743
2.0×10^3	0.198	0.198	8.84	0.516
2.0×10^4	0.210	0.152	29.7	0.395
5.0×10^4	0.237	0.139	53.2	0.361
9.0×10^4	0.300	0.132	90.0	0.343
1.5×10^5	0.893	0.127	346	0.330

of antisymmetric pomeron states is unity ($E = 0$). Correspondingly the odderon built from these states according to (36) has its maximal intercept equal to unity. It is trivial to show that the introduction of a running coupling does not change this result.

Indeed consider the Schroedinger equation (1) for the forward pomeron: $q_1 + q_2 = 0$, $T_1 T_2 = -N_c$. Divided by N_c , in terms of function $\eta(q)$ this equation for the semi-amputated function

$$\phi(q) = \eta(q)\psi_2(q) \quad (40)$$

has the form

$$\begin{aligned} \phi(q) \int \frac{d^2 q_1}{(2\pi)^2} \frac{\eta(q)}{\eta(q_1)\eta(q-q_1)} - 2 \int \frac{d^2 q_1}{(2\pi)^2} \frac{1}{\eta(q-q_1)} \phi(q_1) \\ = E\phi(q). \end{aligned} \quad (41)$$

This equation has an obvious antisymmetric solution

$$\phi(q) = \frac{\mathbf{q}}{\eta(q)}, \quad (42)$$

corresponding to the eigenvalue $E = 0$. Indeed, with (42) the left-hand side of (41) acquires the form

$$\mathbf{q} \int \frac{d^2 q_1}{(2\pi)^2} \frac{1}{\eta(q_1)\eta(q-q_1)} - 2 \int \frac{d^2 q_1}{(2\pi)^2} \frac{\mathbf{q}_1}{\eta(q_1)\eta(q-q_1)}. \quad (43)$$

In the second term, the change of variables $q_1 \rightarrow q - q_1$ transforms the numerator into $\mathbf{q} - \mathbf{q}_1$. The half of the sum of these equal expressions coincides with the first term in (43), so that the left-hand side vanishes. Thus (42) is indeed a solution of (41) with a zero eigenvalue.

5 Conclusions

The introduction of the running coupling via the bootstrap relation advocated for the pomeron a long time ago in [4, 5] allows one to construct in a straightforward manner also an equation for the odderon with a running coupling. Its form admits solutions expressed via antisymmetric pomeron states, similar to the ones found in the fixed coupling case. The maximal value of the intercept for such solutions remains equal to unity as in the fixed coupling case.

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